2.06. Prove $\oint d\vec{r} \times \vec{B} = \int\int_S (\hat{n} \times \vec{\nabla}) \times \vec{B} \, dS$.

Consider a vector $\vec{A} = \vec{B} \times \vec{c}$, where $\vec{c}$ is a constant vector.

Then by Stoke's theorem applied to the vector field $\vec{A}$:

$$\int\int_S \underbrace{\left[\vec{\nabla} \times (\vec{B} \times \vec{c})\right]}_{\vec{\nabla} \times \vec{A}} \cdot d\vec{S} = \oint_C \underbrace{(\vec{B} \times \vec{c})}_{\vec{A}} \cdot d\vec{r}$$

Use identity: $\vec{\nabla} \times (\vec{B} \times \vec{c}) = (\vec{c} \cdot \vec{\nabla}) \vec{B} - \vec{c}(\vec{\nabla} \cdot \vec{B})$
$$- (\vec{B} \cdot \vec{\nabla}) \vec{c} + \vec{B}(\vec{\nabla} \cdot \vec{c}) \quad \left(\begin{array}{c}\text{Prove}\\\text{Take home}\end{array}\right)$$

Since $\vec{c}$ is a constant vector, 3rd and 4th terms of the above vanish.

$$\Rightarrow \quad \int\int_S \left[\vec{\nabla} \times (\vec{B} \times \vec{c})\right] \cdot d\vec{S}$$

$$= \int\int_S \left[(\vec{c} \cdot \vec{\nabla}) \vec{B} - \vec{c}(\vec{\nabla} \cdot \vec{B})\right] \cdot \underbrace{d\vec{S}}_{\hat{n} \, dS}$$

$$= \int\int_S \left[(\vec{c} \cdot \vec{\nabla}) \vec{B} - \vec{c}(\vec{\nabla} \cdot \vec{B})\right] \cdot \hat{n} \, dS$$

Now, consider $\left[(\vec{c} \cdot \vec{\nabla}) \vec{B}\right] \cdot \vec{D}$

$= D_j C_i \partial_i B_j = C_i \partial_i (D_j B_j) - B_j C_i \partial_i D_j$

$$= \vec{C} \cdot \left[\vec{\nabla} \left(\vec{D} \cdot \vec{B} \right) \right] - \vec{B} \cdot \left[\left(\vec{C} \cdot \vec{\nabla} \right) \vec{D} \right]$$

Choose $\vec{D} = \hat{n}$

$$\Rightarrow \left[\left(\vec{C} \cdot \vec{\nabla} \right) \vec{B} - \vec{C} \left(\vec{\nabla} \cdot \vec{B} \right) \right] \cdot \hat{n}$$

$$= \vec{C} \cdot \left[\vec{\nabla} \left(\hat{n} \cdot \vec{B} \right) \right] - \vec{B} \cdot \left[\left(\vec{C} \cdot \vec{\nabla} \right) \hat{n} \right] - \vec{C} \cdot \left[\hat{n} \left(\vec{\nabla} \cdot \vec{B} \right) \right]$$

$$\left(\text{By using above identity} \right)$$

$$= \vec{C} \cdot \left[\vec{\nabla} \left(\vec{B} \cdot \hat{n} \right) - \hat{n} \left(\vec{\nabla} \cdot \vec{B} \right) \right] - \vec{B} \cdot \left[\left(\vec{C} \cdot \vec{\nabla} \right) \hat{n} \right]$$

Next, $\left[\left(\vec{D} \times \vec{\nabla} \right) \times \vec{B} \right]_i$

$$= \epsilon_{ijk} \left(\vec{D} \times \vec{\nabla} \right)_j B_k = \epsilon_{ijk} \epsilon_{jlm} D_l \partial_m B_k$$

$$= \underbrace{\epsilon_{jki} \epsilon_{jlm}}_{\delta_{kl} \delta_{im} - \delta_{km} \delta_{il}} D_l \partial_m B_k \qquad = D_k \partial_i B_k - D_i \partial_k B_k$$

$$= \partial_i \left(D_k B_k \right) - B_k \partial_i D_k - D_i \left(\vec{\nabla} \cdot \vec{B} \right)$$

$$= \partial_i \left(\vec{D} \cdot \vec{B} \right) - D_i \left(\vec{\nabla} \cdot \vec{B} \right) - \vec{B} \cdot \left(\partial_i \vec{D} \right)$$

$$= \left[\vec{\nabla} \left(\vec{D} \cdot \vec{B} \right) \right]_i - \left[\vec{D} \left(\vec{\nabla} \cdot \vec{B} \right) \right]_i - \vec{B} \cdot \left(\partial_i \vec{D} \right)$$

$$\therefore \vec{C} \cdot \left[(\vec{D} \times \vec{v}) \times \vec{B} \right]$$

$$= C_i \left[(\vec{D} \times \vec{v}) \times \vec{B} \right]_i$$

$$= \vec{C} \cdot \left[\vec{v} (\vec{D} \cdot \vec{B}) - \vec{D} (\vec{v} \cdot \vec{B}) \right] - \vec{B} \cdot \left(C_i \, \partial_i \, \vec{D} \right)$$

$$= \vec{C} \cdot \left[\vec{v} (\vec{D} \cdot \vec{B}) - \vec{D} (\vec{v} \cdot \vec{B}) \right] - \vec{B} \cdot \left[(\vec{C} \cdot \vec{v}) \, \vec{D} \right]$$

So, for $\vec{D} = \hat{n}$, the above yields

$$\vec{C} \cdot \left[(\hat{n} \times \vec{v}) \times \vec{B} \right] = \left[(\vec{C} \cdot \vec{v}) \vec{B} - \vec{C} (\vec{v} \cdot \vec{B}) \right] \cdot \hat{n}$$

(Give this as take home assignment)

Therefore we have,

$$\iint_S \left[\vec{v} \times (\vec{B} \times \vec{C}) \right] \cdot d\vec{s} = \iint_S \vec{C} \cdot \left[(\hat{n} \times \vec{v}) \times \vec{B} \right] ds$$

$$= \vec{C} \cdot \iint_S \left[(\hat{n} \times \vec{v}) \times \vec{B} \right] ds$$

$$\left(\text{As } \vec{C} \text{ is a constant vector} \right)$$

On the other hand

$$\oint_C (\vec{B} \times \vec{C}) \cdot d\vec{r} = \oint_C \vec{C} \cdot (d\vec{r} \times \vec{B})$$

$$= \vec{C} \cdot \oint_C d\vec{r} \times \vec{B}.$$

Now since $\vec{c}$ is completely arbitrary, equating the above two one obtains,

$$\iint_S \left[\left(\hat{n} \times \vec{\nabla} \right) \times \vec{B} \right] dS = \oint_C d\vec{r} \times \vec{B}.$$